Smaller alignment index (SALI): a new indicator distinguishing between ordered and chaotic motion

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Abstract

We apply the recently introduced method of the smaller alignment index (SALI) for distinguishing between ordered and chaotic orbits (Skokos 2001) in a 2-dimensional (2D) and a 4-dimensional (4D) symplectic map.

1 Introduction

The distinction between ordered and chaotic motion in a dynamical system is fundamental in a large area of modern science. This distinction is particularly difficult in systems with many degrees of freedom, basically because we cannot visualize their phase space. So, we need fast and accurate tools to give us information about the chaotic or ordered character of an orbit, mainly for dynamical systems with more than three degrees of freedom.

Many methods that try to give an answer to this problem have been developed over the years. The inspection of the consequents of an orbit on a Poincaré surface of section has been used extensively mainly for 2D systems. One of the most common method is the computation of the maximal Lyapunov Characteristic Number (LCN) (Benettin et al. 1976, Froeschlé 1984), which can be applied for systems with many degrees of freedom. Another efficient method is the frequency map analysis developed by Laskar (Laskar 1990, Laskar et al. 1992). In recent years new methods have been developed like the study of spectra of "short time Lyapunov characteristic numbers" (Froeschlé et al. 1993, Lohinger et al. 1993) or "stretching numbers" (Voglis & Contopoulos 1994, Contopoulos et al. 1995) and the "spectral distance" of such spectra (Voglis et al. 1999), as well as the study of spectra of helicity and twist angles (Contopoulos & Voglis 1996, 1997, Froeschlé & Lega 1998). In addition Froeschlé introduced the fast Lyapunov indicator (FLI) (Froeschlé et al. 1997) while Vozikis et al. (2000) proposed a method based on the frequency analysis of "stretching numbers".

Recently a new, fast and easy to compute indicator of the chaotic or ordered nature of orbits, has been introduced: the smaller alignment index (SALI) (Skokos 2001). In the current communication we recall the definitions of the alignment indices and show their effectiveness in distinguishing between ordered and chaotic motion, by applying them in a 2D and a 4D symplectic map.

2 Definition of the alignment indices

Let us consider the n–dimensional phase space of a symplectic map $\mathbf{T}$, an orbit in that space with initial condition $P(0) = (x_1(0),\ x_2(0), \ldots,\ x_n(0))$ and a deviation vector $\xi(0) = (dx_1(0),\ dx_2(0), \ldots,\ dx_n(0))$ from the initial point $P(0)$. The evolution in time of the orbit and the deviation vector are given by the mapping $\mathbf{T}$ and the corresponding tangent map as follows:

$$\begin{array}{rcl} P(N+1) & = & \mathbf{T}P(N) \\ \xi(N+1) & = & (\frac{\partial \mathbf{T}}{\partial P(N)})\xi(N) \end{array} . \qquad (1)$$

We note that in mappings the time is discrete i.e. the number N of the iterations. In order to determine if this orbit is ordered or chaotic we follow the evolution in time of two different initial deviation vectors (e.g. $\xi_1(0)$, $\xi_2(0)$). We define as parallel alignment index, the quantity:

$$d_- \equiv \|\xi_1(N) - \xi_2(N)\| \tag{2}$$

and as antiparallel alignment index, the quantity:

$$d_+ \equiv \|\xi_1(N) + \xi_2(N)\| \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector. It is obvious from the above definitions that when $d_- = 0$ the two vectors coincide and when $d_+ = 0$ the two vectors are opposite.

As explained by Voglis et al. (1999), in 2D maps the ordered motion occurs on an 1D torus, the so–called invariant curve and any two deviation vectors, after a transient period, become tangent to this curve, tending to coincide or become opposite to each other. This means that one of the ALIs tends to zero. A similar behavior appears when the orbit tested is chaotic: any two deviation vectors eventually become tangent to the most unstable nearby manifold and so one of the ALIs tends to zero. The transient phase needed for the vectors to take their final orientation has been studied by Skokos et al. (2001) and Vozikis (2001). If we consider the vectors ξ_1 and ξ_2 to be normalized with norm equal to 1, the two deviation vectors tend to coincide when $d_- \to 0$ and $d_+ \to 2$ and tend to become opposite when $d_- \to 2$ and $d_+ \to 0$. So in 2D maps the smaller alignment index (SALI) tends to zero both for ordered and chaotic orbits following however different time rates (as it is shown in the next section), which allows us to distinguish between the two cases.

On the other hand, in the case of 4D maps the distinction between ordered and chaotic motion is even easier. In 4D maps the ordered motion occurs on a 2D torus, on which any initial deviation vector becomes almost tangent after a short transient period. In general, two different initial deviation vectors become tangent to different directions on the torus producing different sequences of vectors, so that both quantities d_+ and d_- tend to positive values in the interval $(0, 2)$. For chaotic orbits any two initially different deviation vectors tend to coincide to the direction defined by the most unstable nearby manifold and hence coincide to each other, or one vector tends to the opposite of the other. This means than one ALI tends to zero. So, the SALI tends to zero when the orbit is chaotic and to a non-zero value when the orbit is ordered. Thus, the completely different behavior of the SALI helps us distinguish between ordered and chaotic motion in 4D maps.

3 Application of the alignment indices

Following Skokos (2001) we compute the ALIs in some simple cases of ordered and chaotic orbits in symplectic maps with two and four dimensions. In particular we use the 2D map:

$$\begin{array}{rcl} x_1' & = & x_1 + x_2 \\ x_2' & = & x_2 - \nu \sin(x_1 + x_2) \end{array} \quad (\text{mod } 2\pi) \tag{4}$$

and the 4D map:

$$\begin{array}{l} x_1' = x_1 + x_2 \\ x_2' = x_2 - \nu \sin(x_1 + x_2) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\ x_3' = x_3 + x_4 \\ x_4' = x_4 - \kappa \sin(x_3 + x_4) - \mu[1 - \cos(x_1 + x_2 + x_3 + x_4)] \end{array} \quad (\text{mod } 2\pi) \tag{5}$$

which is composed of two 2D maps of the form (4), with parameters ν and κ, coupled with a term of order μ. All variables are given (mod 2π), so $x_i \in [-\pi, \pi)$, for

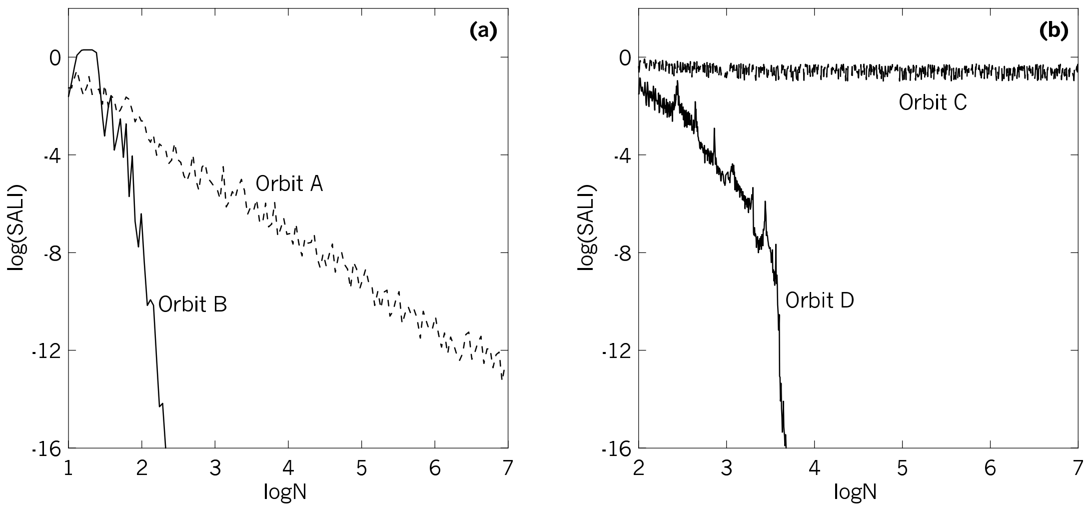

Figure 1: The evolution of the smaller alignment index SALI, with respect to the number N of iterations of the map (a) for the 2D map (4) with $\nu = 0.5$, for the ordered orbit A with initial conditions $x_1 = 2$, $x_2 = 0$ (dashed line) and for the chaotic orbit B with initial conditions $x_1 = 3$, $x_2 = 0$ (solid line) and (b) for the 4D map (5) with $\nu = 0.5$, $\kappa = 0.1$, $\mu = 10^{-3}$, for the ordered orbit C with initial conditions $x_1 = 0.5$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 0$ (dashed line) and for the chaotic orbit D with initial conditions $x_1 = 3$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 0$ (solid line).

$i = 1, 2, 3, 4$. The map (5) is a variant of Froeschlé's 4D symplectic map (Froeschlé 1972). The periodic orbits of map (5) and their bifurcations have been studied by Contopoulos & Giorgilli (1988). Structures in the phase space of this map for small values of the coupling parameter μ were examined in detail by Skokos et al. (1997).

In the case of the 2D map (4) we consider the ordered orbit A with initial conditions $x_1 = 2$, $x_2 = 0$ and the chaotic orbit B with initial conditions $x_1 = 3$, $x_2 = 0$ for $\nu = 0.5$. The initial deviation vectors used are $(1, 0)$ and $(0, 1)$ for both orbits. These vectors eventually coincide in both cases, but on completely different time rates. This is evident from figure 1(a), where the SALI (which coincides with d_- for both orbits) is plotted as a function of the number N of iterations, for the ordered orbit A (dashed line) and the chaotic orbit B (solid line). For the ordered orbit A the SALI decreases as N increases, following a power law and it becomes SALI $\approx 10^{-13}$ after 10^7 iterations, which means that the two deviation vectors almost coincide. On the other hand the SALI of the chaotic orbit B decreases abruptly, reaching the limit of accuracy of the computer (10^{-16}) after about 200 iterations. After that time the two vectors are identical since their coordinates are represented by the same numbers in the computer. So, it becomes evident that the SALI can distinguish between ordered and chaotic motion in a 2D map, since it tends to zero following completely different time rates.

In the case of the 4D map (5) for $\nu = 0.5$, $\kappa = 0.1$ and $\mu = 10^{-3}$ we consider the ordered orbit C with initial conditions $x_1 = 0.5$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 0$ and the chaotic orbit D with initial conditions $x_1 = 3$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 0$. The initial deviation vectors used are $(1, 1, 1, 1)$ and $(1, 0, 0, 0)$. As we see in figure 1(b) the SALI of the ordered orbit C (which coincides with d_-) remains almost constant, fluctuating around SALI ≈ 0.28. On the other hand, the SALI of the chaotic orbit D (which coincides with d_+) decreases abruptly reaching the limit of accuracy of the computer (10^{-16}) after about 4.7 10^3 iterations. After that time the coordinates of the two vectors are represented by opposite numbers in the computer. So, in 4D maps the SALI tends to zero for chaotic orbits, while it tends to a positive non-zero value for ordered orbits. Thus, the different behavior of SALI clearly distinguish between ordered and chaotic orbits.

An advantage of using the ALIs in 4D maps is that usually the chaotic nature of an orbit can be established beyond any doubt. This happens because when the orbit under consideration is chaotic, the SALI becomes equal to zero, in the sense that it reaches the limit of the accuracy of the computer. After that time the two deviation vectors are identical (equal or opposite), since their coordinates are represented by the same or opposite numbers in the computer. Thus they have exactly the same evolution in time and cannot be separated.

A more detailed study of the ALIs and the SALI for several symplectic maps and a comparison with other methods that try to determine the ordered or chaotic nature of an orbit is given in Skokos (2001).

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